Modular Semantics for Open Transition Rules with Negative Premises

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Structural Operational Semantics and Negative Premises

- Structural Operational Semantics specifies a transition (evaluation) relation ($\xrightarrow{l}$) via *inductive rules*.
- Sometimes, authors of process algebras like to use negative premises. E.g.:

$$\frac{x \xrightarrow{l} x'}{x; y \xrightarrow{l} x'; y} \qquad \frac{\{x \stackrel{l}{\nrightarrow}\}_l \quad y \xrightarrow{m} y'}{x; y \xrightarrow{m} y'}$$

- Sometimes negative premises are needed, e.g. certain priority operators inexpressible using just positive premises [Aceto and Ingólfsdóttir(2008)].

Semantics of Systems with Negative Premises?

- No longer a simple inductive definition of provable transitions.
- Potential pitfalls, e.g. rules like $\dfrac{a \stackrel{l}{\nrightarrow}}{a \stackrel{l}{\longrightarrow} b}$
- Various approaches, that of *well-supported proofs* is a popular & powerful notion [Glabbeek(2004)]
- Is *incomplete* for pathological examples like that above
  - neither $\overline{a \stackrel{l}{\nrightarrow}}$ nor $\overline{a \stackrel{l}{\longrightarrow} b}$ are derivable
  - by restricting attention to complete specifications, one achieves a 2-valued TSS

Towards open formulae

- Well-supported proof only works for closed formulas
  - Asserting provability of $s \xrightarrow{l} s'$ or $s \stackrel{l}{\nrightarrow}$ for closed s,s'.
- We wish to extend the notion to open formulae, with hypotheses are variables. e.g.

$$\frac{\{x \stackrel{l}{\nrightarrow}\}_l \qquad \{y \stackrel{l}{\nrightarrow}\}_l \qquad z \xrightarrow{m} z'}{(x;y);z \xrightarrow{m} z'}$$

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- Why?
  - To support (open) *operational laws* via (fh-)bisimulation which remain valid under disjoint extensions [Mosses et al.(2010)Mosses, Mousavi, and Reniers]
  - e.g. $(x;y);z \sim x;(y;z)$

Contribution

A notion of **well-supported proof for open transition rules** satisfying various desirable properties:

- Consistency ($\overline{s \xrightarrow{l} s'}$ and $\overline{s \overset{l}{\nrightarrow}}$ can't both be provable)
- Instantiation closure (if $\overline{s}$ is provable then so is $\overline{\sigma(s)}$)
- Agrees with original notion on closed terms
- Modularity (under disjoint extensions, old proofs remain valid)
- Conservativity (under disjoint extensions, no new proofs of old formulae)

Well-Supported Proofs

Basic Notions

Transition System Specifications have:

- A signature Σ and set of labels L.
- *Formulas* ϕ are of the form $s \xrightarrow{l} s'$ or $s \stackrel{l}{\nrightarrow}$ where s, s' are Σ-terms and $l \in L$.
  - $s \xrightarrow{l} s'$ *denies* $s \stackrel{l}{\nrightarrow}$ and vice-versa.
- A set of deduction rules $\dfrac{H}{s \xrightarrow{l} s'}$ over such formulas.

A *derivation* of a transition rule $\frac{H}{\phi}$ is an inductive proof using rules in D with open leaves/hypotheses (possibly negative) in H.

Ground well-supported proof

A *ground well-supported proof* of $\overline{\phi}$ is a upwardly branching tree labelled by closed formulae and rooted at ϕ, where:

- Positive steps $\dfrac{K}{s \xrightarrow{l} s'}$ are instances of deduction rules
- For negative steps $\dfrac{K}{s \overset{l}{\nrightarrow}}$, it must be the case that:

Each derivation of $\dfrac{N}{s \xrightarrow{l} s'}$, ($N$ negative)
contains some formula which denies a formula in K

Negative steps work by refuting each possible derivation.

Closed-Instance Semantics

- As we will see, the above definition doesn't work for open formulae / transition rules.
- An alternative is *closed-instance semantics*: ϕ holds for open ϕ if all closed instantiations $\sigma(\phi)$ holds.
- But this fails to be *modular*:
  - In a base system with single rule $\frac{x \stackrel{b}{\nrightarrow}}{f(x) \xrightarrow{a} x}$, $\overline{f(x) \xrightarrow{a} x}$ holds.
  - But disjointly adding $\overline{1 \xrightarrow{b} 1}$ invalidates the formula.

Towards open formulae

Example

Consider a TSS with deduction rules $\dfrac{f(x) \stackrel{a}{\nrightarrow}}{g(x) \xrightarrow{a} x}$, $\overline{f(0) \xrightarrow{a} 0}$. *Then:*

- $\overline{f(1) \stackrel{a}{\nrightarrow}}$ *and* $\overline{g(1) \xrightarrow{a} 1}$ *have well-supported proofs.*
- *The derivation* $\overline{f(0) \xrightarrow{a} 0}$ *ensures that neither* $\overline{f(0) \stackrel{a}{\nrightarrow}}$ *nor* $\overline{g(0) \xrightarrow{a} 0}$ *are provable.*
- $\overline{f(x) \stackrel{a}{\nrightarrow}}$ *is provable... shouldn't be, due to the derivation* $\overline{f(0) \xrightarrow{a} 0}$ *which denies an instance of* $f(x) \stackrel{a}{\nrightarrow}$.

Towards open formulae

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$\Rightarrow$ We must consider counterexamples up to substitution: otherwise, $\overline{g(x) \xrightarrow{a} x}$ provable, but $\overline{g(0) \xrightarrow{a} 0}$ unprovable.

Well-supported proofs for open formulas

We next adapt the notion of well-supported proof to open transition rules $\frac{H}{\phi}$ where H is a context:

- H gives assumptions on variables ($x \xrightarrow{l} s$, $x \stackrel{l}{\nrightarrow}$).

Well-supported proofs for open formulas

A *well-supported proof* of $\frac{H}{\phi}$ is a upwardly branching tree labelled by formulae and rooted at ϕ, where:

- Leaves are in H
- Positive steps $\dfrac{K}{s \xrightarrow{l} s'}$ are instances of deduction rules
- For negative steps $\dfrac{K}{s \stackrel{l}{\nrightarrow}}$, it must be the case that:

Each derivation of $\dfrac{C}{\sigma(s) \xrightarrow{l} s'}$, $\quad$ (C negative + vars)
contains a formula denying $\sigma(k)$ for some $k \in K$

(Differences from closed version:
H hypotheses, substitutive counter examples.)

Basic Results

Instantiation Closure

Theorem (Closure under Instantiating Formulae)

Suppose $\frac{\{\psi_i : i \in I\}}{\phi}$ *has a well-supported proof.*
Let σ *be a substitution so each* $\frac{K}{\sigma(\psi_i)}$ *has a well-supported proof.*
Then $\frac{K}{\sigma(\phi)}$ *has a well-supported proof.*

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Then $\dfrac{K}{\sigma(\phi)}$ *has a well-supported proof.*

Proof: Substitution + pasting of proof trees.

Consistency

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In any TSS, it can't be the case that $\overline{s \xrightarrow{l} s'}$ *and* $\overline{s \overset{l}{\nrightarrow}}$ *both have well-supported proofs.*

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Generalisation: Some consistency assumptions on $H \Rightarrow$
can't prove both $\dfrac{H}{s \xrightarrow{l} s'}$ and $\dfrac{H}{s \stackrel{l}{\nrightarrow}}$

Modularity

Disjoint Extensions, Modularity

A *disjoint extension* of a TSS is:

- An extension of the signature Σ with new symbols Σ' and labels
- An extension of D with new rules D', each of which is of the form $\dfrac{S}{f(s_1, \ldots, s_n) \xrightarrow{l} t}$ for $f \in D'$.

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Important property: If π is a well-supported proof of $\frac{H}{\phi}$ in T, then remains so in $T \uplus T_1$.

For positive steps $\dfrac{K}{s \xrightarrow{l} s'}$, simple.

Modularity for Negative Steps

For negative steps we need:

$$\frac{K}{s \stackrel{l}{\not\rightarrow}} \text{ is valid in } T_0 \Rightarrow \text{valid in } T_0 \uplus T_1.$$

i.e. each counterexample proving $\dfrac{C}{\sigma(s) \xrightarrow{l} s'}$ must be denied for $\sigma \in T_0 \uplus T_1$

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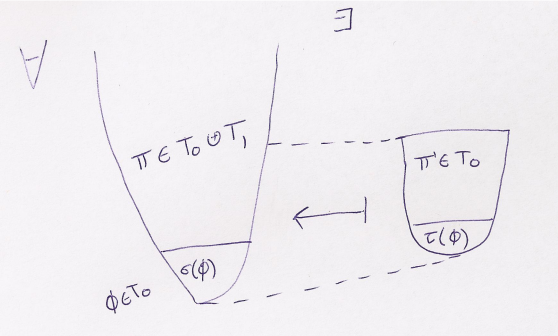

Modularity for well-supported proofs

Theorem (Modularity)

Suppose $T_0 \uplus T_1$ *is a disjoint extension of* T_0 *and let* π *be a well-supported proof for* $\frac{H}{\phi}$ *in* T_0*.*

Then π *is a well-supported proof for* $\frac{H}{\phi}$ *in* $T_0 \uplus T_1$*.*

Conservativity

Source dependence

Now seek to show: in disjoint extensions, no new proofs of old formulae.
Requires *source dependence*:

each variable in a rule can be traced back to a variable in the source of the conclusion (via transitions in the premise)

Ok: $$\frac{x \xrightarrow{l} x'}{x; y \xrightarrow{l} x'; y}$$

Source dependence

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Ok: $\dfrac{x \xrightarrow{l} x'}{x; y \xrightarrow{l} x'; y}$

Example

Consider a TSS $\dfrac{x \xrightarrow{b} 1}{0 \xrightarrow{a} 1}$. *Then* $\overline{0 \xrightarrow{a} 1}$ *not provable.*

Extend by constant 2 *with* $\overline{2 \xrightarrow{b} 1}$. *Then* $\overline{0 \xrightarrow{a} 1}$ *is provable.*

Conservativeness for Disjoint Extensions

Theorem (Conservativeness for Disjoint Extensions)

Let $T_0 \uplus T_1$ be a disjoint extension of T_0, where T_0 is source-dependent, and let $\phi \in T_0$. Let π be a well-supported proof of $\frac{H}{\phi}$ in $T_0 \uplus T_1$. Then π is a well-supported proof of $\frac{H}{\phi}$ in T_0.

Proof: induction using "source dependence measure" for positive steps. For negative steps, uses modularity result to move counterexamples from T_0 to $T_0 \uplus T_1$.

Soundness over Closed-instance Semantics

Theorem

For closed ϕ, if $\overline{\phi}$ has a well-supported proof then it has a ground well-supported proof.

Proof: Follows from the fact that $\overline{\phi}$ has a closed well-supported proof (instantiation closure).

Conservativity over Closed-instance Semantics

Needs source dependence:

Example

Consider TSS T with deduction rule $\dfrac{x \xrightarrow{b} 1}{0 \xrightarrow{a} 1}$.
Then $\overline{0 \overset{a}{\nrightarrow}}$ *has a ground well-supported proof (no valid derivations concluding* $0 \xrightarrow{a} \_$*)*
But no well-supported proof in T.

Conservativity over Closed-instance Semantics

Theorem

In a source dependent system and closed ϕ, if $\overline{\phi}$ has a ground well-supported proof then it has a well-supported proof.

Proof: Follows from the fact that each derivation of $s \xrightarrow{l} s'$ for closed s is closed.

Conclusions

Contribution

Our notion satisfies:

- Consistency ($\overline{s \xrightarrow{l} s'}$ and $\overline{s \stackrel{l}{\nrightarrow}}$ can't both be provable)
- Instantiation closure (if $\overline{s}$ is provable then so is $\overline{\sigma(s)}$)
- Modularity (under disjoint extensions, old proofs remain valid)

Assuming source dependent rules:

- Agrees with original notion on closed terms
- Conservativity (under disjoint extensions, no new proofs of old formulae)

Open Algebraic Laws

Consider an algebraic law, like

$$(x; y); z \sim x; (y; z)$$

As the language is (disjointly) extended, the domain of quantification (x,y,z) increases. Ideal:

- we prove such laws in the "minimal subsystem" containing just the rules for ;
- guaranteed to hold in any extension = any system containing this notion of ;

Fh-bisimulation

To prove such laws, we need to consider a notion of bisimulation for open terms satisfying this modularity property.

fh-bisimulation is such a notion:

- if $s\ R\ t$ and $\dfrac{H}{s \xrightarrow{l} s'}$ then $\dfrac{H}{t \xrightarrow{l} t'}$ with $s'\ R\ t'$

(usual 'step' condition, but under arbitrary hypotheses on variables.)

This notion *is* modular – preserved by disjoint extensions. [Mosses et al.(2010)Mosses, Mousavi, and Reniers]

...with negative premises

- The work here can be used to adapt fh-bisimulation to the negative setting.
- Modularity of the underlying well-supported proofs leads to modularity for the proved equations.
- Another key issue: compositionality (bisimulation as a congruence, via rule formats) [Mousavi et al.(2007)Mousavi, Reniers, and Groote]

PLanCompS vision

- A growing repository of fundamental constructs (like ;) specified independently
- Laws about such constructs can be proved once and for all
  - e.g. associativity/commutativity/unit laws
- Formal semantics can be given in an accessible manner by *translation* into funcons
  - Tool support – e.g. running programs
- Computational effects via the mechanics of *Modular SOS* [Mosses(2004), Churchill and Mosses(2013)]

`www.plancomps.org`

Conclusions

We:

- Extended well-supported proofs to open transition rules
- Proved consistency, instantiation, modularity, conservativity results

Further directions:

- Use these results to support modularity of equational laws
- Consider compositionality of fh-bisimulation based on these notions
- ...

Thank You.

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