Divergence as State in Coinductive Big-Step Semantics

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Why coinduction in operational semantics?

Many interesting languages involve diverging computations

Coinduction provides a tool for reasoning about both converging and diverging computations

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Traditional approach to divergence in big-step semantics introduces rules with duplication

In this talk: Expressing divergence in big-step semantics using a single set of rules without duplication

Divergence in small-step semantics

$\boxed{e \to e'}$

- Inductive interpretation ($\xrightarrow{*}$)

$$\frac{e \to e' \quad e' \xrightarrow{*} e''}{e \xrightarrow{*} e''} \qquad \frac{}{e \xrightarrow{*} e} \qquad e_1 \to e_2 \to \cdots \to e_n$$

Divergence in small-step semantics

$$\boxed{e \to e'}$$

- Inductive interpretation ($\stackrel{*}{\to}$)

$$\frac{e \to e' \quad e' \stackrel{*}{\to} e''}{e \stackrel{*}{\to} e''} \qquad \frac{}{e \stackrel{*}{\to} e} \qquad e_1 \to e_2 \to \cdots \to e_n$$

- Coinductive interpretation ($\stackrel{\infty}{\to}$)

$$\frac{e \to e' \quad e' \stackrel{\infty}{\to}}{e \stackrel{\infty}{\to}} \qquad e \to e_0 \to \cdots \to e_n \to \cdots$$

Big-step semantics for λ-calculus

$$\mathit{Terms} \ni a, b, v ::= c \mid \lambda x.a \mid x \mid a\,b$$

$$\frac{}{c \Rightarrow c}\text{Const} \qquad \frac{}{\lambda x.a \Rightarrow \lambda x.a}\text{Fun}$$

$$\frac{a_1 \Rightarrow \lambda x.b \quad a_2 \Rightarrow v_2 \quad b[x \leftarrow v_2] \Rightarrow v}{a_1\,a_2 \Rightarrow v}\text{App}$$

Inductive interpretation ($\Rightarrow$): the set of pairs (a, v), such that $a \Rightarrow v$ is the conclusion of a finite derivation tree

Divergence in λ-calculus

$$\cdots$$

$$\frac{a_1 \overset{\infty}{\Rightarrow}}{a_1\, a_2 \overset{\infty}{\Rightarrow}}\text{App-l} \qquad \frac{a_1 \Rightarrow \lambda x.b \quad a_2 \overset{\infty}{\Rightarrow}}{a_1\, a_2 \overset{\infty}{\Rightarrow}}\text{App-r}$$

$$\frac{a_1 \Rightarrow \lambda x.b \quad a_2 \Rightarrow v_2 \quad b[x \leftarrow v_2] \overset{\infty}{\Rightarrow}}{a_1\, a_2 \overset{\infty}{\Rightarrow}}\text{App-f}$$

Coinductive interpretation ($\overset{\infty}{\Rightarrow}$): the set of terms a, such that $a \overset{\infty}{\Rightarrow}$ is the conclusion of an infinite derivation tree

$$\omega = \Delta\,\Delta \quad \text{where} \quad \Delta = (\lambda x. x\, x)$$

Inductive relation:

$$\omega \not\Rightarrow v \text{ for any } v$$

Coinductive relation:

$$\dfrac{\dfrac{}{\Delta \Rightarrow \Delta}\text{Lam} \qquad \dfrac{}{\Delta \Rightarrow \Delta}\text{Lam} \qquad x\,x[x \leftarrow \Delta] \overset{\infty}{\Rightarrow}}{\Delta\,\Delta \overset{\infty}{\Rightarrow}}\text{App-f}$$

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Problem: Duplication

Big-step semantics:

- 3 rules
- 3 premises
- 0 duplicate premises

Big-step semantics with divergence:

- 6 rules
- 9 premises
- 3 duplicate premises

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- 6 rules
- 9 premises
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It would be good to avoid those extra rules
– especially when scaling up to larger languages!

Leroy and Grall's coinductive big-step semantics

$$\frac{}{c \stackrel{\text{co}}{\Rightarrow} c}\text{Const} \qquad \frac{}{\lambda x.a \stackrel{\text{co}}{\Rightarrow} \lambda x.a}\text{Fun}$$

$$\frac{a_1 \stackrel{\text{co}}{\Rightarrow} \lambda x.b \quad a_2 \stackrel{\text{co}}{\Rightarrow} v_2 \quad b[x \leftarrow v_2] \stackrel{\text{co}}{\Rightarrow} v}{a_1\, a_2 \stackrel{\text{co}}{\Rightarrow} v}\text{App}$$

Coinductive interpretation ($\Rightarrow$): the set of pairs (a, v), such that $a \stackrel{\text{co}}{\Rightarrow} v$ is the conclusion of a finite or infinite derivation tree

$$\omega = \Delta\,\Delta \quad \text{where} \quad \Delta = (\lambda x.x\,x)$$

$$\dfrac{\dfrac{}{\Delta \overset{\mathsf{co}}{\Rightarrow} \Delta}\,\text{Lam} \qquad \dfrac{}{\Delta \overset{\mathsf{co}}{\Rightarrow} \Delta}\,\text{Lam} \qquad x\,x[x \leftarrow \Delta] \overset{\mathsf{co}}{\Rightarrow} v}{\Delta\,\Delta \overset{\mathsf{co}}{\Rightarrow} v}\,\text{App}$$

$$\omega = \Delta\,\Delta \quad \text{where} \quad \Delta = (\lambda x. x\, x)$$

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Problem: Diverging terms that do not co-evaluate

$$\frac{\omega \overset{\infty}{\Rightarrow}}{\omega\,(0\,0) \overset{\infty}{\Rightarrow}}\,\text{App-l} \quad \text{but} \quad \frac{\omega \overset{\text{co}}{\Rightarrow} \lambda x.b \quad (0\,0) \overset{\text{co}}{\not\Rightarrow} v_2 \quad b[x \leftarrow v_2] \overset{\text{co}}{\Rightarrow} v}{\omega\,(0\,0) \overset{\text{co}}{\not\Rightarrow} v}\,\text{App}$$

Problem: Diverging terms that do not co-evaluate

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There are also terms which do not contain stuck sub-terms, but still do not co-evaluate [Leroy and Grall, 2009].

Divergence as state

$$\mathit{Terms} \ni a, b, v ::= c \mid \lambda x.a \mid x \mid a\, b$$

$$\boxed{a \quad \Rightarrow v}$$

$$\frac{}{c \quad \Rightarrow c}\ \delta\text{-Const} \qquad \frac{}{\lambda x.a \quad \Rightarrow \lambda x.a}\ \delta\text{-Fun}$$

$$\frac{a_1 \quad \Rightarrow \lambda x.b \qquad a_2 \quad \Rightarrow v_2 \qquad b[x \leftarrow v_2] \quad \Rightarrow v}{a_1\, a_2 \quad \Rightarrow v}\ \delta\text{-App}$$

Divergence as state

$$\mathit{Terms} \ni a, b, v ::= c \mid \lambda x.a \mid x \mid a\,b \qquad \mathit{Div} \ni \delta ::= \downarrow \mid \uparrow$$

$$\boxed{a_{/\delta} \Rightarrow v_{/\delta}}$$

$$\frac{}{c_{/\downarrow} \Rightarrow c_{/\downarrow}}\ \delta\text{-Const} \qquad \frac{}{\lambda x.a_{/\downarrow} \Rightarrow \lambda x.a_{/\downarrow}}\ \delta\text{-Fun}$$

$$\frac{a_{1/\downarrow} \Rightarrow \lambda x.b_{/\delta} \quad a_{2/\delta} \Rightarrow v_{2/\delta'} \quad b[x \leftarrow v_2]_{/\delta'} \Rightarrow v_{/\delta''}}{a_1\,a_{2/\downarrow} \Rightarrow v_{/\delta''}}\ \delta\text{-App}$$

$$\frac{}{a_{/\uparrow} \Rightarrow b_{/\uparrow}}\ \text{Div}$$

$$\omega = \Delta\,\Delta \quad \text{where} \quad \Delta = (\lambda x.x\,x)$$

Inductive interpretation:

$$\omega_{/\downarrow} \not\Rightarrow v_{/\delta} \text{ for any } v, \delta$$

Coinductive interpretation: for any v,

$$\dfrac{\dfrac{}{\Delta_{/\downarrow} \overset{\text{co}}{\Rightarrow} \Delta_{/\downarrow}}\,\delta\text{-Lam} \qquad \dfrac{}{\Delta_{/\downarrow} \overset{\text{co}}{\Rightarrow} \Delta_{/\downarrow}}\,\delta\text{-Lam} \qquad \Delta\,\Delta_{/\downarrow} \overset{\text{co}}{\Rightarrow} v_{/\uparrow}}{\Delta\,\Delta_{/\downarrow} \overset{\text{co}}{\Rightarrow} v_{/\uparrow}}\,\delta\text{-App}$$

Divergence as state covers all diverging terms!

In contrast to Leroy and Grall's coinductive big-step semantics:

$$\text{for all } e, \quad e_{/\downarrow} \overset{\mathsf{co}}{\Rightarrow} v_{/\uparrow} \quad \text{iff} \quad e \overset{\infty}{\Rightarrow}$$

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Divergence as state:
Automatic rule transformation
allowing us to reason about divergence
while avoiding the duplication problem

Problem: Noise in the coinductive interpretation

Convergence and divergence are coinductively indistinguishable

$$\dfrac{\dfrac{}{\Delta_{/\downarrow} \overset{\text{co}}{\Rightarrow} \Delta_{/\downarrow}}\,\delta\text{-Lam} \qquad \dfrac{}{\Delta_{/\downarrow} \overset{\text{co}}{\Rightarrow} \Delta_{/\downarrow}}\,\delta\text{-Lam} \qquad \Delta\,\Delta_{/\downarrow} \overset{\text{co}}{\Rightarrow} v_{/\downarrow}}{\Delta\,\Delta_{/\downarrow} \overset{\text{co}}{\Rightarrow} v_{/\downarrow}}\,\delta\text{-App}$$

Noise cancellation

$$\frac{e_{/\downarrow} \overset{\text{CO}}{\Rightarrow} e'_{/\uparrow}}{e \rightsquigarrow \bot} \qquad \frac{e_{/\downarrow} \Rightarrow v_{/\downarrow}}{e \rightsquigarrow v}$$

Noise cancellation

$$\frac{e_{/\downarrow} \overset{\text{co}}{\Rightarrow} e'_{/\uparrow}}{e \rightsquigarrow \bot} \qquad \frac{e_{/\downarrow} \Rightarrow v_{/\downarrow}}{e \rightsquigarrow v}$$

For constructive alternative, see Nakata and Uustalu's work on trace-based coinductive semantics

Summary

Pros:

- Single set of rules, avoids the duplication problem
- More expressive than Leroy and Grall's coinductive big-step semantics

Summary

Pros:

- Single set of rules, avoids the duplication problem
- More expressive than Leroy and Grall's coinductive big-step semantics

Cons:

- Convergence and divergence are coinductively indistinguishable
- Slightly less expressive than traditional divergence predicates $\stackrel{\infty}{\Rightarrow}$